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Equations of motion in linearised gravity: II Run-away sources

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Abstract. Following an approach described in an earlier paper, we study the Robinson–Trautman fields in linearised gravity having run-away sources. We solve the linearised vacuum Einstein–Maxwell field equations when the run-away source in the background Minkowskian space–time is charged. Functions of integration are determined by the requirement that terms be excluded from the field of the particle which are singular on pairs of null-rays emanating into the future from events on the world-line of the particle in the background space–time. In contradistinction we show that, for an uncharged run-away source, one or other, but not both, of the 'directional' singularities which occur in the field of the particle can be removed.

1. Introduction

This paper is a sequel to the preceding paper (Hogan and Imaeda 1979a) which will hereafter be referred to as I.

Since the Robinson–Trautman (1962) solutions of the Einstein and Einstein–Maxwell field equations closely resemble the Liénard–Wiechert solutions of Maxwell's equations, it is natural to examine whether the source of a Robinson–Trautman field can, in a technical sense, perform run-away motion. This type of motion is predicted by the Lorentz–Dirac equation of motion (see Synge 1965) of a point charge subject to its own electromagnetic field and in the absence of an external electromagnetic field. A similar type of motion has been suggested by Havas and Goldberg (1962) for a point mass moving in its own gravitational field. However, their linearised field for the particle is a retarded field which does not satisfy the Sommerfeld outgoing radiation conditions in the form given by Trautman (1958), and, as a consequence, they have radiation falling on the particle from infinity (see Hogan 1974a).

In this paper we study in detail the electromagnetic and gravitational fields of a run-away charge in the linear approximation. Directional singularities, i.e. singularities along future null-rays emanating from events on the world-line of the particle, which arise in a natural way can be removed by choices of functions of integration. We conclude that the Robinson–Trautman solutions include the field of a run-away charged mass. On the other hand we show that the field of a run-away mass has an irremovable directional singularity, and hence the Robinson–Trautman solutions do not contain an acceptable field for a run-away mass. It may well be that there exists no acceptable solution of Einstein's vacuum field equations describing the gravitational field of such a mass.

There has been some previous work, from a different point of view, on the run-away charge problem. This is mentioned in I together with a brief description of the distinction in approaches.

The outline of the paper is as follows. In § 2 we introduce the linearised Robinson–Trautman form of the Einstein–Maxwell vacuum field equations. This is accomplished by expanding quantities in a small parameter (the mass of the source, in a technical sense) from their Minkowskian values, and yields a line-element which is that of Minkowskian space–time plus a small first-order perturbation. The perturbation is singular on an unspecified time-like world-line in the background Minkowskian space–time. In § 3 we specify this world-line to be the history of a run-away particle and solve the linearised field equations derived in the previous section. Finally, in § 4 we discuss briefly the case of a run-away mass.

2. Linearisation

Our starting point is the Robinson–Trautman form for the line-element,

$$ds^2 = 2r^2P^{-2}\,d\zeta\,d\bar{\zeta} - 2\,dr\,d\sigma - h\,d\sigma^2, \tag{2.1}$$

where r, σ are real coordinates, and ζ, $\bar{\zeta}$ are complex coordinates, the bar indicating complex conjugation. The function P is independent of r, but in general depends upon ζ, $\bar{\zeta}$ and σ, while h is a function of all four coordinates. The corresponding electromagnetic four-potential is given by the one-form

$$\Phi = -F\,d\sigma, \tag{2.2}$$

where F is a function of all four coordinates. The vacuum Einstein–Maxwell field equations are satisfied by (2.1) and (2.2), provided (Robinson and Trautman 1962, Robinson 1973, private communication with P A Hogan)

$$h = K - 2Hr - 2M/r + e^2/r^2, \qquad F = e(1/r - w), \tag{2.3a}$$

$$K = \Delta \ln P \qquad (\Delta = 2P^2\partial^2/\partial\zeta\partial\bar{\zeta}), \tag{2.3b}$$

$$H = \partial(\ln P)/\partial\nu, \tag{2.3c}$$

$$M = m + 2e^2w, \tag{2.3d}$$

$$\Delta w = -2H, \qquad \tfrac{1}{4}\Delta K = \dot{M} - 3HM + e^2N, \tag{2.3e}$$

$$N = 2P^2(\partial w/\partial\zeta)\partial w/\partial\bar{\zeta}, \tag{2.3f}$$

where $w = w(\zeta, \bar{\zeta}, \sigma)$, the 'dot' indicates differentiation with respect to σ, and we have chosen m and e to be constants. We will take m and e to be the mass and charge respectively of our linearised source. We shall assume that m and e^2 are small of first order, writing† $m = O_1$, $e^2 = O_1$, and expand P and w above in the form

$$P = \underset{0}{P}(1+Q) + O_2, \tag{2.4a}$$

† We choose units for which $c = G = 1$ and Gaussian units for which $4\pi = 1$. Hence we should have a parameter l (say), having the dimensions of length, so that $ml^{-1} = O_1$ and $e^2l^{-2} = O_1$. In the sequel the specialisation to a run-away source will involve a parameter b which is the acceleration of the source when $\sigma = 0$. We would choose $l = b^{-1}$.

where $Q = O_1$ and

$$w = \underset{0}{w} + \underset{1}{w} + O_2, \tag{2.4b}$$

with $\underset{1}{w} = O_1$. Substituting these into (2.3*b*), (2.3*c*) and (2.3*f*) yields

$$H = \underset{0}{H} + \underset{1}{H} + O_2 \tag{2.5a}$$

$$K = \underset{0}{K} + \underset{1}{K} + O_2, \tag{2.5b}$$

$$N = \underset{0}{N} + \underset{1}{N} + O_2, \tag{2.5c}$$

with

$$\underset{0}{H} = \partial(\ln \underset{0}{P})/\partial\sigma, \qquad \underset{0}{K} = \underset{0}{\Delta} \ln \underset{0}{P}, \qquad \underset{0}{N} = 2\underset{0}{P}^2(\partial\underset{0}{w}/\partial\zeta)\partial\underset{0}{w}/\partial\bar{\zeta}, \tag{2.6}$$

where $\underset{0}{\Delta} = 2\underset{0}{P}^{-2}\partial^2/\partial\zeta\partial\bar{\zeta}$, while

$$\underset{1}{H} = \dot{Q}, \tag{2.7a}$$

$$\underset{1}{K} = \underset{0}{\Delta} Q + 2\underset{0}{K} Q, \tag{2.7b}$$

$$\underset{1}{N} = 4\underset{0}{P}^2[(\partial\underset{0}{w}/\partial\zeta)\partial\underset{1}{w}/\partial\bar{\zeta} + (\partial\underset{0}{w}/\partial\bar{\zeta})\partial\underset{1}{w}/\partial\zeta + Q(\partial\underset{0}{w}/\partial\zeta)\partial\underset{0}{w}/\partial\bar{\zeta}]. \tag{2.7c}$$

We shall assume that the subscript zero refers to the Minkowskian values of the quantities defined above. In I we have shown that when the Minkowskian line-element is written in the form (2.1) it is given explicitly by

$$ds^2 = 2r^2\underset{0}{P}^{-2}\, d\zeta\, d\bar{\zeta} - 2\, dr\, d\sigma - (1 - 2\underset{0}{H}r)\, d\sigma^2, \tag{2.8}$$

where

$$\underset{0}{P} = \lambda^4(1 + \tfrac{1}{2}\zeta\bar{\zeta}) - \lambda^3(1 - \tfrac{1}{2}\zeta\bar{\zeta}) - (3/\sqrt{2})(\lambda^1 - i\lambda^2) - (\bar{\zeta}/\sqrt{2})(\lambda^1 + i\lambda^2), \tag{2.9a}$$

$$\underset{0}{H} = \partial(\ln \underset{0}{P})/\partial\sigma = -\mu^j k_j. \tag{2.9b}$$

The notation here is the same as in I. If X^i, $i = 1, 2, 3, 4$, are rectangular Cartesian coordinates and time in this background Minkowskian space–time, then $r = 0$ is a time-like world-line with equation $X^i = x^i(\sigma)$. Its tangent or four-velocity is $\lambda^i = dx^i/d\sigma$, and its four-acceleration is $\mu^i = d\lambda^i/d\sigma$. k^i is tangent to the future-pointing generators of the null-cones at every event on $r = 0$ and is normalised so that $\lambda_j k^j = -1$. For a detailed derivation of (2.8) and (2.9) the reader is refered to I. In this notation the Liénard–Wiechert four-potential is given by (see Synge 1965) the one-form

$$\underset{0}{\Phi} = (e/r)\lambda_j\, dX^j. \tag{2.10}$$

Using the formulae (3.2) of I for differentiating retarded quantities, this can be written

$$\underset{0}{\Phi} = -e(1/r + \mu^j k_j)\, d\sigma - d(e \ln r). \tag{2.11}$$

The second term here can be removed by a gauge transformation, and thus an equivalent form to (2.10) is

$$\underset{0}{\Phi} = -e(1/r - \underset{0}{w})\, \mathrm{d}\sigma, \tag{2.12a}$$

$$\underset{0}{w} = -\mu^j k_j = \underset{0}{H}. \tag{2.12b}$$

With the values of $\underset{0}{P}$ and $\underset{0}{w}$ given above it is easy to see that

$$\underset{0}{\Delta} \ln \underset{0}{P} = 1, \qquad \underset{0}{\Delta}\underset{0}{w} + 2\underset{0}{w} = 0. \tag{2.13}$$

The first of these implies that $\underset{0}{K}$ appearing in (2.5*b*) and (2.6) is unity, while the second is the first of (2.3*e*) in zeroth order. The first-order perturbations in (2.5) are now calculated using (2.3*e*). These equations read

$$\underset{0}{\Delta}\underset{0}{w} = 4Q\underset{0}{H} - 2\underset{1}{H} + \mathrm{O}_2, \tag{2.14a}$$

$$\tfrac{1}{4}\underset{0}{\Delta}\underset{1}{K} = 2e^2\underset{0}{\dot{w}} - 3\underset{0}{H}(m + 2e^2\underset{0}{w}) + 2e^2\underset{0}{P}^2(\partial\underset{0}{w}/\partial\zeta)\partial\underset{0}{w}/\partial\bar{\zeta} + \mathrm{O}_2. \tag{2.14b}$$

The procedure for solving these equations is as follows. We begin by specifying the time-like world-line $r = 0$ in the Minkowskian background space–time. Thence we know $\underset{0}{P}$, $\underset{0}{H}$ and $\underset{0}{w}$. These are substituted into the right-hand side of (2.14*b*) and one solves for $\underset{1}{K}$. This value of $\underset{1}{K}$ is used in (2.7*b*) to obtain Q. Knowing Q we determine $\underset{1}{H}$ from (2.7*a*). These values of Q and $\underset{1}{H}$ are finally substituted into (2.14*a*) and one solves for $\underset{1}{w}$. The calculation is then complete, with the metric tensor and four-potential known with an O_2 error, i.e. in the linear approximation. In carrying out this programme certain functions of integration will occur which are utilised to remove 'directional' singularities in the linearised Weyl and Maxwell tensors. These singularities, by their nature, are extraneous to the field of a simple pole particle.

3. Run-away charge

We specify the time-like world-line in the background Minkowskian space–time to have four-velocity components (Synge 1965)

$$\lambda^1 = \lambda^2 = 0, \qquad \lambda^3 = \sinh[(b/\alpha)(\mathrm{e}^{\alpha\sigma} - 1)], \qquad \lambda^4 = \cosh[(b/\alpha)(\mathrm{e}^{\alpha\sigma} - 1)], \tag{3.1}$$

where b and α are constants. We shall find that α is positive, and thus, mindful of a singularity developing in the infinite future and the corresponding breakdown of the approximation, we confine our study to a future-bounded time interval $-\infty < \sigma < \sigma_0$ for some small $\sigma_0 > 0$. The constant b is easily seen to be the value of the four-acceleration component μ^3 when $\sigma = 0$. The constant α, which will be determined later, occurs in the equation of motion, which is obtained from (3.1) to be

$$\mu^i = \alpha^{-1}(\nu^i - \mu^j \mu_j \lambda^i), \tag{3.2}$$

where $\nu^i = \mathrm{d}\mu^i/\mathrm{d}\sigma$. Equation (3.1) is the four-velocity of a particle accelerating to the speed of light (running away) as $\sigma \to \infty$, if $\alpha > 0$, along the X^3 axis. If (3.1) is substituted into (2.9) we find

$$\underset{0}{P}=k_1(\tfrac{1}{2}\zeta\bar{\zeta}+k_2^2),\qquad \underset{0}{H}=a(\tfrac{1}{2}\zeta\bar{\zeta}-k_2^2)/(\tfrac{1}{2}\zeta\bar{\zeta}+k_2^2),\tag{3.3}$$

$$k_1=\lambda^3+\lambda^4=\mathrm{e}^{\chi},\qquad k_2=\lambda^3-\lambda^4=-\mathrm{e}^{-\chi},\tag{3.4}$$

where $a=b\,\mathrm{e}^{\alpha\sigma}$, $\chi=(b/\alpha)(\mathrm{e}^{\alpha\sigma}-1)$ and $k_1k_2=-1$. As we have already noted in I, for a particle moving along the X^3 axis, from the ten Killing vectors of the Minkowskian background, the rotations about the X^3 axis, generated by the vector field

$$\mathrm{i}(\zeta\partial/\partial\zeta-\bar{\zeta}\partial/\partial\bar{\zeta}),\tag{3.5}$$

have a special significance for the model we are constructing here. We again make the reasonable assumption that this symmetry be preserved in the linear approximation. This will be guaranteed if we henceforth require functions to depend on ζ and $\bar{\zeta}$ in the combination $\zeta\bar{\zeta}$ as in (3.3). We find it more convenient, however, to introduce in place of $\zeta\bar{\zeta}$ the new variable

$$\underset{0}{\xi}=(\tfrac{1}{2}\zeta\bar{\zeta}-k_2^2)/(\tfrac{1}{2}\zeta\bar{\zeta}+k_2^2).\tag{3.6}$$

The first equation to solve is (2.14b). Written in terms of $\underset{0}{\xi}$ and using (2.12b) and (3.3) it takes the form

$$\partial[(1-\underset{0}{\xi}^2)\partial\underset{1}{K}/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi}=12e^2a^2(1-3\underset{0}{\xi}^2)+4a(2e^2\alpha-3m)\underset{0}{\xi}+\mathrm{O}_2.\tag{3.7}$$

This can be integrated directly to give

$$\begin{aligned}\underset{1}{K}=6e^2a^2\underset{0}{\xi}^2-2a(2e^2\alpha-3m)\underset{0}{\xi}-A(\sigma)&\\ +[C(\sigma)+2a(2e^2\alpha-3m)]\tfrac{1}{2}\ln[(1+\underset{0}{\xi})/(1-\underset{0}{\xi})]+\mathrm{O}_2,&\end{aligned}\tag{3.8}$$

where A, C are functions of integration. At this stage it is useful to calculate the tetrad components of the linearised Weyl tensor and the linearised Maxwell tensor. A natural null-tetrad to use is given by the one-forms

$$\begin{aligned}&m_i\,\mathrm{d}x^i=rP^{-1}\,\mathrm{d}\bar{\zeta}, &&\bar{m}_i\,\mathrm{d}x^i=rP^{-1}\,\mathrm{d}\zeta,\\ &l_i\,\mathrm{d}x^i=-\mathrm{d}r-\tfrac{1}{2}h\,\mathrm{d}\sigma, &&k_i\,\mathrm{d}x^i=-\mathrm{d}\sigma.\end{aligned}\tag{3.9}$$

However, one can simplify the resulting expressions for the tetrad components of the Weyl and Maxwell tensors, and, in fact, remove the dependence on $\underset{1}{w}$ from them by using a tetrad related to (3.9) by a null-rotation (see Janis and Newman 1965). The specific null-rotation is

$$k^i\to k^i,\qquad m^i\to m^i+gk^i,\qquad l^i\to l^i+g\bar{m}^i+\bar{g}m^i+g\bar{g}k^i\tag{3.10}$$

where

$$g=(r\zeta/\underset{0}{P})(a+\partial\underset{1}{w}/\partial\underset{0}{\xi}+aQ)=ar\zeta\underset{0}{P}^{-1}+\mathrm{O}_1\tag{3.11}$$

In terms of this new null-tetrad, and using the notation of Newman and Penrose (1962), the tetrad components of the Weyl and Maxwell tensors are given respectively by

$$\psi_0=\mathrm{O}_2,\qquad \psi_1=\mathrm{O}_2,\tag{3.12a}$$

$$\psi_2 = -(1/r^3)(m + 2e^2 a\underset{0}{\xi}) + e^2/r^4 + \mathrm{O}_2, \tag{3.12b}$$

$$\psi_3 = (1/2r^2)(\bar{\zeta}/\underset{0}{P})[\partial\underset{1}{K}/\partial\underset{0}{\xi} - 6a(m + 2e^2 a\underset{0}{\xi})] + \mathrm{O}_2 \tag{3.12c}$$

$$\begin{aligned}\psi_4 = {} & (1/r^2)(\bar{\zeta}^2/\underset{0}{P}^2)(\tfrac{1}{2}\partial^2\underset{1}{K}/\partial\underset{0}{\xi} - 6e^2a^2) \\ & + (1/r)(\bar{\zeta}^2/\underset{0}{P}^2)[2a\partial\underset{1}{K}/\partial\underset{0}{\xi} - 6a^2(m + 2e^2a\underset{0}{\xi}) - \partial(\partial\underset{1}{H}/\partial\underset{0}{\xi} + 2aQ)/\partial\underset{0}{\xi}] + \mathrm{O}_2,\end{aligned} \tag{3.12d}$$

$$\Phi_0 = 0, \tag{3.12e}$$

$$\Phi_1 = -e/2r^2 + \mathrm{O}_2, \qquad \Phi_2 = \mathrm{O}_2. \tag{3.12f}$$

We note the following useful result: the transformation

$$Q \to Q + \tfrac{1}{2}A(\sigma) + B(\sigma, \underset{0}{\xi}), \qquad \underset{1}{H} \to \dot{Q} + \tfrac{1}{2}\dot{A} + \dot{B}, \qquad \underset{1}{K} \to \underset{1}{K} + A \tag{3.13}$$

leaves (3.12) invariant provided B satisfies

$$\partial(\partial\dot{B}/\partial\underset{0}{\xi} + 2aB)/\partial\underset{0}{\xi} = 0. \tag{3.14}$$

Also the field equations (2.7*a*), (2.7*b*) and (2.14*b*) are invariant under (3.13) provided

$$\underset{0}{\Delta}B + 2B = 0. \tag{3.15}$$

The field equation (2.14*a*) for $\underset{1}{w}$ is not left invariant by (3.13), but since $\underset{1}{w}$ does not appear in (3.12) we have the result that (3.13) subject to (3.14) and (3.15) constitutes a *gauge transformation.*

On substituting the expression (3.8) for $\underset{1}{K}$ into ψ_3 in (3.12*c*) we find

$$\psi_3 = (1/2r^2)(\bar{\zeta}/\underset{0}{P})\{-4e^2a\alpha + [C + 2a(2e^2\alpha - 3m)](1 - \underset{0}{\xi}^2)^{-1}\} + \mathrm{O}_2. \tag{3.16}$$

This expression is not only singular at $r = 0$, but is also singular at $\underset{0}{\xi} = \pm 1$. On account of (2.9*b*), (3.3) and (3.6), $\underset{0}{\xi} = \pm 1$ corresponds to

$$\mu^i k_i = \pm b\ \mathrm{e}^{\alpha\sigma}. \tag{3.17}$$

Thus for each constant value of σ, i.e. on each future null-cone with vertex on $r = 0$, (3.17) picks out a pair of diametrically opposed future-pointing null-rays or generators, k^i, on which the field component (3.16) is singular. We exclude this singularity from (3.16), since the field of a simple pole particle should only be singular on $r = 0$, by choosing

$$C = -2a(2e^2\alpha - 3m). \tag{3.18}$$

Making this substitution in $\underset{1}{K}$ we pass to (2.7*b*). In terms of the variable $\underset{0}{\xi}$ this reads

$$\partial[(1 - \underset{0}{\xi}^2)\partial Q/\partial\underset{0}{\xi}]/\partial\underset{0}{\xi} + 2Q = 6e^2a^2\underset{0}{\xi}^2 + C\underset{0}{\xi} - A + \mathrm{O}_2. \tag{3.19}$$

Integrating by a standard technique (see Bateman 1918) we obtain

$$Q = \tfrac{3}{2}e^2a^2(1-\underset{0}{\xi}^2) - \tfrac{1}{6}C\underset{0}{\xi}\ln(1-\underset{0}{\xi}^2) - \tfrac{1}{2}A - B(\sigma, \underset{0}{\xi}) + \mathrm{O}_2,$$

$$B(\sigma, \underset{0}{\xi}) = \beta(\sigma)\mathrm{P}_1(\underset{0}{\xi}) + \gamma(\sigma)\mathrm{Q}_1(\underset{0}{\xi}), \tag{3.20}$$

where β, γ are functions of integration, and $\mathrm{P}_1(\underset{0}{\xi})$, $\mathrm{Q}_1(\underset{0}{\xi})$ are $l = 1$ Legendre functions of the first and second kind. From this one calculates $\underset{1}{H}$ in (2.7a) using $\dot{\underset{0}{\xi}} = a(1-\underset{0}{\xi}^2)$, $\dot{a} = \alpha a$ and, on account of (3.18), $\dot{C} = \alpha C$. We then calculate ψ_4 from (3.12d) to find

$$\psi_4 = (1/r)(\bar{\zeta}^2/\underset{0}{P}^2)\{aC - 6ma^2 + 6e^2a^2\alpha + [2\dot{\gamma} - \tfrac{1}{3}\alpha C\underset{0}{\xi}(3-\underset{0}{\xi}^2)](1-\underset{0}{\xi}^2)^{-2} + \mathrm{O}_2. \tag{3.21}$$

The final term here is non-singular at $\underset{0}{\xi} = \pm 1$ only if $\gamma = \gamma_0$, a constant, and $C = 0$. The latter is a remarkable result, for if we return to (3.18) we find that we have determined α to be

$$\alpha = 3m/2e^2, \tag{3.22}$$

and the equation of run-away motion (3.2) must be precisely the Lorentz–Dirac equation (see Synge 1965).

The non-vanishing (modulo an O_2 error) tetrad components of the Weyl and Maxwell tensors are finally given by

$$\psi_2 = -(1/r^3)(m + 2e^2a\underset{0}{\xi}) + e^2/r^4 + \mathrm{O}_2, \qquad \psi_3 = -(3ma/r^2)\bar{\zeta}/\underset{0}{P} + \mathrm{O}_2,$$

$$\psi_4 = (3ma^2/r)\bar{\zeta}^2/\underset{0}{P}^2 + \mathrm{O}_2, \qquad \Phi_1 = -e/2r^2 + \mathrm{O}_2. \tag{3.23}$$

The function Q in (2.4a) is given by (3.20) with $C = 0$, $\underset{1}{K}$ by (3.8) without the logarithmic term, and $\underset{1}{H}$ by $\dot{Q}$. We could then obtain $\underset{1}{w}$ from (2.14a), but it does not contribute to the metric or the field in the linear approximation. The linearised Weyl tensor is Petrov type II, while the Maxwell tensor is algebraically general. We notice that the functions of integration A, β, γ_0 remaining in Q, $\underset{1}{H}$, $\underset{1}{K}$ do not appear in (3.23). It is not surprising then that they can be removed by a gauge transformation. The relevant gauge transformation is given by (3.13). The condition (3.14) reduces to $\dot{\gamma} = 0$ for $B(\sigma, \underset{0}{\xi})$ given by (3.20), while this value for $B(\sigma, \underset{0}{\xi})$ clearly satisfies (3.15).

The quantities (3.23) are all non-singular on $\underset{0}{\xi} = \pm 1$. They are singular on $r = 0$. We note that $-\infty < \sigma < \sigma_0$, so that the singularity in the limit $\sigma \to \infty$ is not allowed to develop. As $\sigma \to -\infty$ the fields (3.23) become the linearised gravitational and electric fields of a static charge, i.e. the Reissner–Nordstrom solution in the linear approximation.

4. Discussion

The final form for the line-element of a run-away charged mass in the linear approximation is given by (2.1), (2.3a) and (2.3d) with

$$
\begin{aligned}
&P=-2k_2(1-\underset{0}{\xi})^{-1}[1+\tfrac{3}{2}e^2a^2(a-\underset{0}{\xi}^2)]+O_2,\\
&H=a\underset{0}{\xi}+\tfrac{9}{2}ma^2(1-\underset{0}{\xi}^2)-3e^2a^3\underset{0}{\xi}(1-\underset{0}{\xi}^2)+O_2,\\
&K=1+6e^2a^2\underset{0}{\xi}^2+O_2, \qquad M=m+2e^2a\underset{0}{\xi}+O_2.
\end{aligned} \tag{4.1}
$$

$\underset{0}{\xi}$ is given by (3.6), k_2 by (3.4), and $a=b\,e^{\alpha\sigma}$, where $-\infty<\sigma<\sigma_0$. The four-potential is given by (2.12) with sufficient accuracy.

It should be noted that, having determined the constant α in the run-away equation of motion (3.2) to have the value stated in (3.22), we arrive at the Lorentz–Dirac equation of motion *without* an infinite self-energy term (see e.g. Hogan 1974*b*).

If we take as source, in the background Minkowskian space–time, a run-away *uncharged* mass, then solving equations (2.3) with $e=0$, in the manner described in I, one finds that the tetrad components of the linearised Riemann tensor are now (using (3.12) with $e=0$)

$$
\begin{aligned}
&\psi_0=\psi_1=O_2, \qquad \psi_2=-m/r_3+O_2,\\
&\psi_3=O_2, \qquad \psi_4=-(1/r)(\bar{\zeta}^2/P^2)S(\underset{0}{\xi})/(1-\underset{0}{\xi}^2)^2+O_2,
\end{aligned} \tag{4.2}
$$

with $a=b\,e^{\alpha\sigma}$ as in (3.3) and

$$
S=2ma\alpha\underset{0}{\xi}(1-\underset{0}{\xi}^2)+4ma\alpha\underset{0}{\xi}-2\dot{\gamma}(\sigma), \tag{4.3}
$$

where $\dot{\gamma}(\sigma)$ is the derivative of a function of integration. This situation is quite different from the case of the run-away charged mass discussed in § 3, for if one chooses $\dot{\gamma}=2ma\alpha$ in (4.3) one removes the singularity in ψ_4 at $\underset{0}{\xi}=+1$, while if one chooses $\dot{\gamma}=-2ma\alpha$ in (4.3) one removes the singularity in ψ_4 at $\underset{0}{\xi}=-1$. The value of α is left undetermined. Hence one has an irremovable 'directional' singularity in the linearised field of a run-away uncharged mass. We therefore conclude that the Robinson–Trautman family of solutions does not contain an acceptable solution describing the field of a run-away neutral mass, in complete contrast to the case of a run-away charged mass.

It is well-known within the Lorentz covariant framework of classical electrodynamics that a charged particle undergoing self-interaction will perform run-away motion (cf Synge 1965). We have an example of this here in the linearised Einstein–Maxwell theory. There is no external field. As we have pointed out in I, the need for an external field to drive the particle is thought to manifest itself in the occurrence of conical singularities in the two-surfaces $r=$ constant, $\sigma=$ constant. In the case discussed in the present paper one can show that no such singularities occur (this is examined in the following paper (Hogan and Imaeda 1979b)).

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